1) The line L passes through the point (5, -10) and is perpendicular to the line $5y-4x=15$. Determine the equation of the line L. [All the responses are in slope-intercept form, so we'll aim for that …]

To write an equation of a line we need a point (Got it !) and the slope. The slope is going to be all the work here – all we know about the slope of *L* (the new line) is that *L* is perpendicular to the given line (AND that perpendicular lines have slopes that are negative reciprocals).

So the steps are:
=> Find the slope of the given (or original) line. We'll solve for *y* …

$$5y-4x=15$$

$$5y=4x+15$$ The original line has a slope of $\frac{4}{5}$. Our new line *L* has a slope of $-\frac{5}{4}$.

$$y=\frac{4}{5}x+3$$

=> So the equation of the line we want is $y=-\frac{5}{4}x+b$. How do we find *b*?

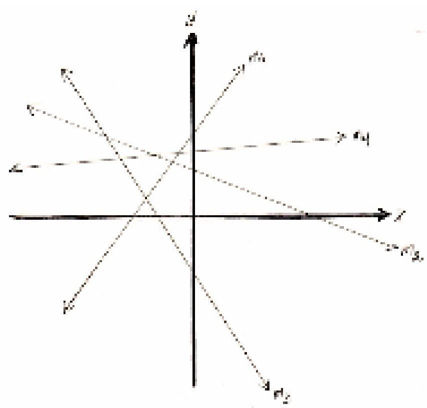
=> This is when we use the point (5, -10). If our equation works for all the points (x, y) on the line, then it must also be true for (5, -10). So we can substitute for *x* and *y* in our equation.

$$-10=-\frac{5}{4}(5)+b => -10=\frac{-25}{4}+b => \frac{-15}{4}=b$$

=> So $y=-\frac{5}{4}x-\frac{15}{4}$.

Note: 68% correct! The most popular incorrect answers had the 'y-intercept' of 3. That's the *y*-intercept of the original line! Why would you think the *y*-intercepts of the given line AND line *L* are the same? Important question – figure out WHY you chose a response with a *y* intercept of 3. Learn to evaluate your own work …. Remember – you are not trying 'just to get by' - you want to master this!

2) List the slopes in increasing order:



They are hard to see when they are all jumbled together like this! But "increasing order" means in "number line order" – from most negative to most positive SLOPE.

86% of my students were correct on this one. I confess I'm not sure what thinking led to the wrong answers – at least the most popular wrong answer. I would love for someone to let me know ….

What you need to know to answer this: lines with positive slope slant up, lines that slant down have negative slope. Steeper slants are for slopes that are more negative or more positive. This is one of those 100% questions … everyone should get it next time!

3) Find the value(s) of M so that the line through the points (M, 9) and (-9, M) is parallel to the line through the points (3, 4) and (M, 1).

This looks like a slope question. Actually, knowing slopes let's you set up the quadratic – then MOST of the problem is an algebra problem. If you don't set the slopes up correctly, though, you're dead in the water.

The key word here is **parallel**. That tells us – in an algebra/precalc class – that we are going to use the fact that the slopes of those two lines are exactly the same. So we find the slopes of the line segments between the two pairs of points:

$$m_1 = \frac{\Delta y}{\Delta x} = \frac{M-9}{-9-M} = \text{(could also write)} = \frac{9-M}{M-(-9)}$$

$$m_2 = \frac{\Delta y}{\Delta x} = \frac{1-4}{M-3} = \frac{-3}{M-3} \text{(could also write this the other way)}$$

These two slopes have to be the same.

Set them equal to each other …

$\frac{M-9}{-9-M} = \frac{-3}{M-3}$. Since we have **fraction=fraction** we can cross-multiply (or find common denominators). $(M-9)(M-3) = (-9-M)(-3) \Rightarrow M^2 - 12M + 27 = 27 + 3M$. **Now** we have a quadratic equation, so we bring everything to one side (so it equals zero)

$M^2 - 15M = 0 \Rightarrow M(M-15) = 0 \Rightarrow M = 0,15$. Checking both answers shows they are both correct, so the correct answer is "M has two values only".

Only 45% of my students were correct. If you got this one wrong, take out a sheet of graph paper (so you can draw an accurate graph) and plot the 4 points you get for each solution (well, (3,4) is common to both solutions). Draw the line segments in – I want you to see what is happening!

And yes, the solution is involved. There are three basic steps (slope, multiply out, solve quadratic) and our goal is that you execute each one nearly flawlessly. BUT more important than that is to develop enough facility with the algebra to not fear problems like this.

4) Identify the center and radius of the circle represented by the following equation.

$$2x^2 + 2y^2 - 16x + 4y = 6$$

Solution 1

To directly solve this problem, you would want to 'complete the squares'. Why? We learned that the equation that describes the points on a circle is:

$$(x-h)^2 + (y-k)^2 = r^2.$$

If you multiply out the two squared terms on the left, you notice that you have *x* terms and a number, as well as *y* terms and a number. So, we're going to our terms above following those 'guidelines'.

$$2x^2 - 16x + 2y^2 + 4y = 6 \Rightarrow 2(x^2 - 8x) + 2(y^2 + 2y) = 6$$

We want to 'complete the square' of the expressions in parentheses. Recall that to do that (as long as the coefficient of the squared terms is 1) we will take the number in front of *x* (or *y*); divide by 2; and square that. That represents the number we'll add to each **parentheses**.

$$\frac{-8}{2} = -4; \quad (-4)^2 = 16♥$$

$$\frac{+2}{2} = 1; \quad (1)^2 = 1♥$$

so we want our equation above to look like

$$2(x^2 - 8x + 16) + 2(y^2 + 2y + 1) = 6 + ? + ?$$

What numbers did we add to the right? We added 16 to the left parentheses – **but** we are multiplying that whole parentheses by **2** so we are really adding **32**. We added 1 to the right parentheses, but we are also multiplying that by **2**, so we really added **2**. Our equation now looks like:

$$2(x^2 - 8x + 16) + 2(y^2 + 2y + 1) = 6 + 32 + 2 = 40$$

Comparing to our model above, we don't want the factors of 2 on the left and, dividing left and right by 2 yields our nearly final version: $(x^2 - 8x + 16) + (y^2 + 2y + 1) = 20$

Changing the parentheses to our squared notation, we get:

$2(x-4)^2+2(y+1)^2=\left(\sqrt{20}\right)^2$ So our center is at (4, -1), and the radius is $\sqrt{20}=\sqrt{4}\bullet\sqrt{5}=2\sqrt{5}$

The correct center of the circle (alone) will let you choose the correct answer with certainty.

Solution 2

$(x-h)^2+(y-k)^2=r^2$ is the equation of a circle. You notice that

* our problem is multiplied out; so we'll do that with our equation of a circle above.
* the 'model' equation above has only coefficients of *1* before everything, so we'll divide our problem's equation by 2

$$(x-h)^2+(y-k)^2=r^2 \Rightarrow x^2-2hx+h^2+y^2-2ky+k^2=r^2$$

$$2x^2+2y^2-16x+4y=6 \Rightarrow x^2+y^2-8x+2y=3$$

Comparing these two 'patterns' tells us that $-2h=-8$ and $-2k=+2$; so $h=+4;\quad k=-1$

And that's our center!

Solution 3

This is the brutal approach. Take each solution and figure out what the equation of **that** circle would be. And compare (carefully) with the equation given in the problem.

Unfortunately, this problem would be difficult to solve if you did not remember the model equation of a circle ($(x-h)^2+(y-k)^2=r^2$). I would encourage you to, again, get a piece of graph paper, and try creating a table of values for some circles. Begin with a center at (0, 0), and a radius of 1, and our model becomes: $(x)^2+(y)^2=1^2=1$. What values of *x* and *y* make this true? Well, when *x* is +1, *y* has to be 0; when *x* is 0, *y* can be either +1 or -1; and so on. When *x* = .5, use a calculator and plot some points, and see what is happening. Pictures can lead much more easily to understanding (and retention!).

5) Determine the value(s) of w such that the points (5, w) and (13, 4) are 10 units apart.

How do we determine how far apart two points are (i.e. the **distance** between them)? We use the **distance formula** which is a simple application of the Pythagorean Theorem.

$$d=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \Rightarrow 10=\sqrt{(13-5)^2+(4-w)^2}. \text{ Squaring both sides ...}$$

$$100=(13-5)^2+(4-w)^2=8^2+\left(16-8w+w^2\right)=80-8w+w^2$$

It's a quadratic, so we'll solve that in the usual manner …

$$w^2-8w-20=0 \Rightarrow (w-10)(w+2)=0 \Rightarrow w=-2,+10$$

Carefully checking each possible solution shows that they are both, correct, and that gives us our correct answer.

This is a pretty standard question using the distance formula, which I would expect all of you to have seen before. That 61% of you were correct on this one was, therefore, a disapoointment.

6) Rationalize the numerator $\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$.

$$\frac{\sqrt{x+h}-\sqrt{x}}{h}=\frac{\left(\sqrt{x+h}-\sqrt{x}\right)}{h}\bullet\frac{\left(\sqrt{x+h}+\sqrt{x}\right)}{\left(\sqrt{x+h}+\sqrt{x}\right)}=\frac{(x+h)-(x)}{h\left(\sqrt{x+h}+\sqrt{x}\right)}=\frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}=\frac{1}{\left(\sqrt{x+h}+\sqrt{x}\right)}$$

7) Simplify the following expression to a single fraction.

$$\frac{\frac{1}{b}-\frac{5}{a}}{\frac{1}{b^2}-\frac{25}{a^2}}.$$

Solution 1

I really dislike the denominators in these multi-level fractions, and I know you do too. So we're going to find something to multiply by that will eliminate all the denominators. We can **always** do this - the trick is to multiply by the smallest thing we can get away with. I notice that the bottom denominators are the worst, so I'll multiply by a^2 and b^2. In other words, by a^2b^2. So …

$$\frac{\left(\frac{1}{b}-\frac{5}{a}\right)}{\left(\frac{1}{b^2}-\frac{25}{a^2}\right)}\bullet\frac{\frac{a^2b^2}{1}}{\frac{a^2b^2}{1}}=\frac{\left(\frac{a^2b^2}{b}-\frac{5a^2b^2}{a}\right)}{\left(\frac{a^2b^2}{b^2}-\frac{25a^2b^2}{a^2}\right)}=\frac{\left(a^2b-5ab^2\right)}{\left(a^2-25b^2\right)}.$$ Progress, but obviously we have to simplify (because we don't yet match the answer choices, for one thing).

$$\frac{\left(a^2b-5ab^2\right)}{\left(a^2-25b^2\right)}=\frac{ab(a-5b)}{(a-5b)(a+5b)}=\frac{ab}{(a+5b)}$$

Another example of why we need to be able to factor (so we can identify the correct answer).

8) Simplify. Write the answer with positive exponents only. $\left(\frac{27x^2y^{-3}}{8x^{-4}y^3}\right)^{1/3}$.

Success in the application of the exponent rules is expected at this stage. The difficulty in writing a solution is that I will apply the exponent rules in a certain order … but it will not be the **only** order. So – your work might very well be different – will almost always be different – but you should still get the correct answer! Over one third of the class still struggles with this however ….

Rule: $(ab)^m=\left(a^mb^m\right)$ and $\left(\frac{a}{b}\right)^m=\left(\frac{a^m}{b^m}\right)$. To eliminate the exponent outside the parentheses.

$$\left(\frac{27x^2y^{-3}}{8x^{-4}y^3}\right)^{1/3}=\left(\frac{27^{1/3}\left(x^2\right)^{1/3}\left(y^{-3}\right)^{1/3}}{8^{1/3}\left(x^{-4}\right)^{1/3}\left(y^3\right)^{1/3}}\right)$$

Rule: $\left(a^m\right)^n=a^{mn}$. Raising a power to another exponent allows you to multiply the exponents – why?

$$\left(\frac{27^{1/3}\left(x^2\right)^{1/3}\left(y^{-3}\right)^{1/3}}{8^{1/3}\left(x^{-4}\right)^{1/3}\left(y^3\right)^{1/3}}\right)=\left(\frac{27^{1/3}\left(x^{2/3}\right)\left(y^{-1}\right)}{8^{1/3}\left(x^{-4/3}\right)\left(y^1\right)}\right)$$

Rule: $\frac{a^m}{a^n}=a^{m-n}=\frac{1}{a^{n-m}}$. Dividing one power by another requires us to subtract exponents – why?

$$\left(\frac{27^{1/3}\left(x^{2/3}\right)\left(y^{-1}\right)}{8^{1/3}\left(x^{-4/3}\right)\left(y^1\right)}\right)=\left(\frac{27^{1/3}}{8^{1/3}}\right)\left(\frac{x^{2/3}}{x^{-4/3}}\right)\left(\frac{y^{-1}}{y^1}\right)=\left(\frac{27^{1/3}}{8^{1/3}}\right)\left(x^2\right)\left(\frac{1}{y^2}\right)=\left(\frac{27^{1/3}}{8^{1/3}}\right)\left(\frac{x^2}{y^2}\right)$$

And at this point I can choose the correct answer. But let's finish …

Rule: $a^{1/n}=\sqrt[n]{a}$. This is the n^{th} root of *a* – the number I multiply *n* times to get *a*.

$$\left(\frac{27^{1/3}}{8^{1/3}}\right)\left(\frac{x^2}{y^2}\right)=\frac{3}{2}\left(\frac{x^2}{y^2}\right)$$

9) Reduce the following fraction to lowest terms.

$$\frac{3x^2(x+1)^3-3(x^3+4)(x+1)^2}{(x+1)^6}$$

The general strategy is to combine the numerator and leave the denominator factored. But if we combine **this** numerator, we'll get degrees of *x* that we can't factor. So we'll try and factor things first – and then combine what's left.

$$\frac{3x^2(x+1)^3-3(x^3+4)(x+1)^2}{(x+1)^6}=\frac{\left(3(x+1)^2\right)\left[x^2(x+1)-(x^3+4)\right]}{(x+1)^6}=\frac{\left(3(x+1)^2\right)\left[x^3+x^2-(x^3+4)\right]}{(x+1)^6}=$$

$$\frac{\left(3(x+1)^2\right)\left[x^2-4\right]}{(x+1)^6}=\frac{\left(3(x+1)^2\right)(x-2)(x+2)}{(x+1)^6}$$

And we'll cancel the common **factors**: $\dfrac{\left(3(x+1)^2\right)(x-2)(x+2)}{(x+1)^6}=\dfrac{3(x-2)(x+2)}{(x+1)^4}$

This was a dreadful question, with one of the incorrect choices ($\dfrac{3(x^2-x^3-4)}{(x+1)}$) being selected 50% more often than the correct answer. Again … you may only cancel **common factors** and <u>**never**</u> common terms. Understand why this answer is impossible! Now!

10) Simplify your answer. $\dfrac{x^2-16}{2x^2+10x+8}\bullet\dfrac{x^3+1}{x^2-13x+36}$.

To simplify something like this, we need to factor everything we can – because we can only cancel common factors! So …

$\dfrac{x^2-16}{2x^2+10x+8}\bullet\dfrac{x^3+1}{x^2-13x+36}=\dfrac{(x-4)(x+4)}{2(x+4)(x+1)}\bullet\dfrac{x^3+1}{(x-9)(x-4)}$ and let's say we've forgotten how to factor the sum of two cubes. Our next step would be to cancel the **factors** common to the top and bottom.

$\dfrac{(x-4)(x+4)}{2(x+4)(x+1)}\bullet\dfrac{x^3+1}{(x-9)(x-4)}=\dfrac{1}{2(x+1)}\bullet\dfrac{x^3+1}{(x-9)}$. This does not match any of the choices but we can eliminate 3 of the 5 choices already. We can use long division [$(x+1)\overline{)x^3+1}$] to find the other factor, and the answer will be $\dfrac{1}{2}\bullet\dfrac{x^2-x+1}{(x-9)}=\dfrac{x^2-x+1}{2(x-9)}$.

11) Solve $4+\left|\dfrac{1}{4}x-3\right|>7$.

There are several types of 'bad things' in algebra that we want to isolate on one side of an equation – it's a very common strategy. And it's a strategy that carries over to other disciplines as well. The idea is that IF you can examine this thing all by itself, then you can learn more about it. Here's a short list: **the *x* term** (when solving an algebra 1 linear equation), **radicals**, all the terms in a **quadratic expression**, and here the **absolute value expression**. So …

$4+\left|\dfrac{1}{4}x-3\right|>7\Rightarrow\left|\dfrac{1}{4}x-3\right|>3$. The expression inside the absolute value bars can be either positive or negative. Right? That gives us two **different** situations we have to consider.

=> **If the expression is positive**

… then we know that we can 'remove' the absolute value bars. So

$\left|\frac{1}{4}x-3\right|>3 \Rightarrow \frac{1}{4}x-3>3 \Rightarrow \frac{1}{4}x>6 \Rightarrow x>24$. This is the interval written $(24,\infty)$

=> **If the expression is negative**

… then we know that the expression in the abs. val. bars would have to be LESS THAN -3. So –

$\left|\frac{1}{4}x-3\right|>3 \Rightarrow \frac{1}{4}x-3<-3 \Rightarrow \frac{1}{4}x<0 \Rightarrow x<0$. This is the interval written $(-\infty,0)$

Notice that another strategy here is to just try *x* values in the interval. When we use the number '24' as the endpoint to one of these intervals, understand clearly what we are saying: Something **different** happens on either side of that number. So … try '23' and then try '25' as values for *x*. Does one make the inequality true, and the other make the inequality false? IF NOT, then ignore all those choices …. If you are checking your work, no one should pick the answer with the 3/2 in it ….

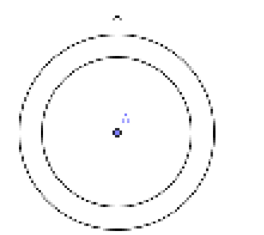
12) Solve for *x*. $\frac{1}{x+2}=x-6$.

Wow …. If I rewrite this as $\frac{1}{x+2}=\frac{x-6}{1}$ then I can just cross-multiply. $1=(x+2)(x-6)$. Now I know I have to multiply out and collect all the terms on one side …

$1=(x+2)(x-6) \Rightarrow 1=x^2-4x-12 \Rightarrow 0=x^2-4x-13$ which doesn't factor so I must use the quadratic formula. Just substitute in *a* = 1; *b* = -4; *c* = -13 …

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(-13)}}{2(1)}=\frac{4\pm\sqrt{16+52}}{2}=\frac{4\pm\sqrt{68}}{2}=\frac{4\pm\sqrt{4}\sqrt{17}}{2}=2\pm\sqrt{17}.$$

13) A circular garden is surrounded by a path of uniform width. If the path has area 56π sq. ft. and the radius of the garden is 5 ft. Find the width of the path.



So we have two concentric circles, and we want the area left between them. In other words, we want the area of the larger circle MINUS the area of the smaller circle. The radius of the smaller circle is 5 ft.; the radius of the larger circle is 5 ft. PLUS whatever the width of the path is – let's call the width of the path *x*. So the radius of the large circle is 5+*x*.

Therefore, the area of the path is:

$\pi(5+x)^2-\pi 5^2=\pi(25+10x+x^2)-\pi\bullet 25=25\pi+10x\pi+x^2\pi-25\pi=10x\pi+x^2\pi$ We are told that this is equal to 56π so (do you smell the algebra coming?)

$10x\pi+x^2\pi=56\pi \Rightarrow x^2\pi+10x\pi-56\pi=0 \Rightarrow \pi(x^2+10x-56)=0$ So we factor the quadratic and find our *x* values. $(x^2+10x-56)=0 \Rightarrow (x+14)(x-4)=0$ and *x* must be 4 (the width of the path can't be negative).

14) Solve for *x*: $x+\sqrt{x-4}=4$. What is the **sum** of the solutions?

This is a standard question 'device' to reduce the 2 solutions to one number we can list in our choices. It helps to reduce the 'luck' factor …. As we mentioned in #11 we wish to learn more about what's inside the radical, so we have to get the radical all by itself.

$x+\sqrt{x-4}=4 \Rightarrow \sqrt{x-4}=4-x$ Then we can square both sides OF THE EQUATION …

$\left(\sqrt{x-4}\right)^2=(4-x)^2 \Rightarrow (x-4)=16-8x+x^2 \Rightarrow x^2-9x+20=0 \Rightarrow (x-5)(x-4)=0$ **but remember** when we square both sides we may be introducing a **"fake"** solution. Check each one in the original: *x*=4 turns out to be the only correct solution. So "4" is the answer!

15) Solve for *x*: $\frac{2x-1}{x+5}>0$.

These **rational function inequalities** is one of the places we use the '**sign chart**' technique. The number all by itself is zero so we can proceed…. Recall – we identify all the *x* values that make the rational function zero, and we identify all the *x* values that make the rational function **undefined** (essentially the *x*s that make the denominator zero…).

$2x-1=0 \Rightarrow x=\frac{1}{2}; \quad x+5=0 \Rightarrow x=-5$. These points are drawn on our number line, and we test each interval to determine the sign in that interval (the idea is that the sign can only change from positive to negative (or vice-versa) if the expression goes THROUGH zero.

$f(x)=\frac{2x-1}{x+5}:$ ______________________________

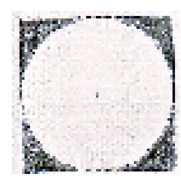
-5 +1/2 $f(-6)=\frac{-13}{-1}=13>0;\ f(0)=\frac{-1}{5}<0;\ f(1)=\frac{1}{6}>0$

-6 *0* *1*

So our function is positive when *x* < -5 and *x* > 1/2. $(-\infty,-5)\cup(1/2,\infty)$.

16) Solve for *x*: $x^2-9\geq 0$. You could use the same technique as #15. But seriously ….
$x^2-9\geq 0 \Rightarrow x^2\geq 9$. So *x* has to be bigger than +3 or less than -3. There is only one answer choice that has two intervals.

17) A circle is inscribed in a square. If the length of the side of the square is *t*-units, determine the area of the shaded region in terms of *t*.



An old geometry trick … but not much of a trick. If the side of the square is *t* then so is the diameter of the circle. So … the radius of the circle must be *t* / 2. Subtract the smaller area from the larger ….

$t^2-\pi\left(\frac{t}{2}\right)^2=t^2-\pi\frac{t^2}{4}=t^2\left(1-\frac{\pi}{4}\right)$.